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Determination of the temperature dependent thermophysical properties from temperature responses measured at medium's boundaries

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Abstract

A numerical method is presented to determine the temperature dependent thermal conductivity and heat capacity from the temperature measured at boundaries in a medium. The undetermined thermophysical properties are denoted as the unknown variables in a set of nonlinear equations, which are formulated from the measured temperature and the calculated temperature at the medium's boundaries. Then, an iterative process is used to solve the set of nonlinear equations. Two sensors are needed to measure the temperature in the medium. The results show that the speed of convergence is considerably fast because the number of iterations to approach a satisfied solution is under seven times. The close agreement between the exact values and the estimated results is made to confirm the validity and accuracy of the proposed method.

1. Introduction

The determination of the thermophysical properties from the measured temperature profiles is a coefficient inverse problem of heat conduction [1]. This is an imperative problem because the magnitude of the thermophysical properties has a significant influence on the analysis of temperature distribution, heat flow rate, and thermal instability problems. In most of the practical engineering problems, thermophysical properties are temperature dependent and lead the heat equation to a nonlinear form. The determination of the temperature dependent thermophysical properties is more difficult than that of the temperature independent thermophysical properties such as constant type, temporal-dependent type, or spatial-dependent type [2–6]. Alifanov and Mikhailov [7], Tervola [8], Scarpa et al. [9], Huang et al. [10], Lam and Yeung [11], have proposed methods to estimate the temperature dependent thermal conductivity alone. However, only few works have been done to estimate the temperature dependent thermal conductivity and heat capacity simultaneously [12–15]. The problem of estimating two properties simultaneously is an interesting topic because the maximal information can be gathered from measured temperatures in only one experiment. Artyukhin [12] developed an iterative algorithm to estimate the thermal conductivity and the heat capacity simultaneously but no numerical simulation was tested. Huang and Ozisik [13] used a direct integration approach to resolve the problem but only the linear type of properties was investigated. Huang and Yan [14] used a conjugate gradient method to estimate both properties simultaneously and the simulated temperatures are

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Nomenclature

$C(T)$	estimated heat capacity
$\mathbf{J}$	error function
$k(T)$	estimated thermal conductivity
k_m, C_n	undetermined coefficient
$\tilde{m}$, $\tilde{n}$	upper bound of the indices of coefficients
N_t	number of the temporal steps
N^{meas}	number of the spatial measurements
T	temperature
t	temporal coordinate
Δt	increment of temporal domain
X_m	sensitivity function of T with respect to k_m
X_n	sensitivity function of T with respect to C_n
x	spatial coordinate
$\mathbf{x}_0$	vector of the initial guess

Greek symbols

$\mathbf{\Phi}$	vector constructed from Φ
ϕ_m, φ_n	basis function
Φ	calculated temperature minus measured temperature
Φ_c	calculated temperature
Φ_{meas}	measured temperature
$\mathbf{\Psi}$	sensitivity matrix
Δ	increment of the search step
ϵ	value of the stopping criterion
σ	standard deviation of measurement error
λ	random number
κ	parameter to regularize the temperature variable
τ	regularized temperature variable

Subscripts

i, j, u, v	indices
meas	measured value
m, n	indices for undetermined coefficients
λ	iterative step

Superscripts

-	dimensional parameters
^	exact estimated function
meas	measured value
exact	exact value

measured at all discretized grids with a minor measurement error. Sawaf et al. [15] used a Levenberg–Marquardt method to estimate the linearly temperature dependent thermal conductivity and heat capacity in a two-dimensional orthotropic medium.

An accurate estimation of two thermophysical properties is more difficult than that of one thermophysical property. For example, Huang and Yan [14] used eleven sensors to estimate two properties simultaneously while Huang et al. [10] used only two sensors to estimate one property. In the first study, a maximum measurement error 0.1% induces a 4.101% and 1.026% average relative errors in the thermal conductivity and the heat capacity, respectively, when two properties are estimated simultaneously. In the second study, a 2% average relative measurement error leads to a 6% average relative error when the thermal conductivity is estimated alone. From the results, it shows that the estimation of two properties needs so many sensors that the experiment becomes difficult and impracticable. As well, the estimation of two properties is very sensitive to the measurement errors, which reduce the accuracy of the estimation. Furthermore, the above studies employ the nonlinear least-squares formulation to determine the thermophysical properties. The method takes the squares of the difference between the measured temperature and the calculated temperature from the medium, and that increases the nonlinearity of the problem. In other words, the problem is formulated in a more complicated form and makes the problem more pronounced.

The purpose of this research is to propose an efficient and stable method to estimate the temperature dependent thermal conductivity and heat capacity simultaneously. In the proposed approach, the determination of the thermophysical properties includes two phases: the process of direct analysis and the process of inverse analysis. In the direct analysis process, the thermophysical properties are assumed as the known values and then directed to solve the temperature field of the heat conduction equation through a numerical method. Solutions from the above process are integrated with the available temperature measured at the sensors' locations. Thus, a set of nonlinear equations is formulated for the process of the inverse estimation. In the inverse analysis process, an iterative method is used to guide the exploring points systematically to approach to the undetermined thermophysical properties. Then, the intermediate properties are substituted for the unknown properties in the following analysis. As such, several iterations are needed for obtaining the undetermined thermophysical properties. In the present research, the proposed method formulates the problem from the difference between the calculated temperature and the measured one directly. Therefore, the inverse formulation derived from the proposed method is simpler than that from the nonlinear least-squares method. Furthermore, only two thermocouples are needed to be allocated at the boundaries, which is much easier than that of the eleven-pointed measurement in the past research.

2. The proposed method to estimate the thermal conductivity and the heat capacity simultaneously

An iterative algorithm is used to simultaneously estimate the temperature dependent thermal conductivity and heat capacity of a homogenous medium using two-pointed measurements in a transient heat conduction experiment. Some treatments are needed in the process of solving the inverse problem. They are the direct problem, the sensitivity problem, the operational algorithm, and the stopping criterion. The direct problem is used to determine the temperature distribution and the sensitivity problem is used to find the search step in the inverse problem. The operational algorithm is used to fulfill the process of the inverse analysis when the solutions of the direct problem and the sensitivity problem are available. Finally, the stopping criterion is shown to stop the iterative process.

The problem in this research is to estimate the thermal conductivity and the heat capacity simultaneously from the two-pointed measurements. The limitation of the proposed method is that a function form is needed to represent the undetermined thermophysical property. Suppose that the thermophysical properties $k(T)$ and $C(T)$ are expressed as the following linear forms corresponding to k_m and C_n in the problem domain:

$$k(T)=\sum_{m=1}^{\bar{m}}k_m\phi_m(T) \tag{1}$$

$$C(T)=\sum_{n=1}^{\bar{n}}C_n\varphi_n(T) \tag{2}$$

where $\phi_m(T)$ and $\varphi_n(T)$ are any first derivative continuous functions in the problem domain and k_m and C_n are the undetermined coefficients. $\bar{m}$ and $\bar{n}$ are integers.

Therefore, the inverse problem is to determine the values of the coefficients k_m and C_n.

2.1. The direct problem

Consider a slab with $\bar{L}$ thickness and temperature dependent thermal conductivity and heat capacity. This slab originally has a uniformed temperature field. At a beginning time, $\bar{t}=0$, a heat flux $\bar{q}_1$ is applied to the front surface at $\bar{x}=0$ and another heat flux $\bar{q}_2$ is applied to the back surface at $\bar{x}=\bar{L}$. The temperature field over the slab is $\bar{T}_0$ when $\bar{t}=0$. A dimensionless mathematical formation is written in the following formulation:

$$\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right]=C(T)\frac{\partial T}{\partial t}\quad 0<x<1,\quad t>0 \tag{3}$$

$$T(x,0)=1\quad 0\le x\le 1\quad t=0 \tag{4}$$

$$-k(T)\frac{\partial T}{\partial x}=q_1\quad x=0\quad t>0 \tag{5}$$

$$-k(T)\frac{\partial T}{\partial x}=q_2\quad x=1\quad t>0 \tag{6}$$

where the following dimensionless quantities are defined as:

$$x=\frac{\bar{x}}{\bar{L}}\quad T=\frac{\bar{T}}{\bar{T}_0}\quad k=\frac{\bar{k}}{\bar{k}_r}\quad C=\frac{\bar{C}}{\bar{C}_r}\quad q=\frac{\bar{L}\bar{q}}{\bar{k}_r\bar{T}_0}$$
$$t=\frac{\bar{k}_r}{\bar{\rho}\bar{C}_r}\frac{\bar{t}}{\bar{L}^2}$$

$\bar{T}_0$, $\bar{k}_r$, and $\bar{\rho}\bar{C}_r$ refer to the nonzero reference temperature, the thermal conductivity, and the heat capacity per unit volume, respectively. $k(T)$ and $C(T)$ are the unknown temperature dependent thermal conductivity and heat capacity. From the physical viewpoint, the values of thermal conductivity and heat capacity must be strongly positive, i.e. $k(T)>0$ and $C(T)>0$. If $k(T)$ and $C(T)$ are estimated simultaneously, the boundary condition and the measured method need to satisfy the following two requirements to insure the uniqueness of the solution. First, the heat flux must be known at one boundary at least and it must not be vanished. Second, two sensors are needed to measure the temperature histories at least and they can be located at the boundaries with the prescribed heat flux [12].

The direct problem is used to generate the simulated temperature when the values of $k(T)$ and $C(T)$ are specified. It is a nonlinear problem because the coefficients in Eq. (3) are functions of temperature. Therefore, a finite difference method is used to solve the direct problem iteratively when the initial and boundary conditions are given. Then, the results from the direct analysis can be substituted into the sensitivity equation and lead to a sensitivity analysis.

2.2. The sensitivity problem

In the proposed method, an iterative algorithm is adopted to solve the inverse problem in that the sensitivity analysis is necessary to decide the search step in each iteration. After Eqs. (1) and (2) are substituted into Eqs. (3)–(6), the derivative $\partial/\partial k_m$ and $\partial/\partial C_n$ are taken at both sides of equations. Then, we have

$$k(T)\frac{\partial^2 X_m}{\partial x^2}+\frac{\partial k(T)}{\partial T}\frac{\partial T}{\partial x}\frac{\partial X_m}{\partial x}+\frac{\partial^2 T}{\partial x^2}\frac{\partial k(T)}{\partial k_m}+\left(\frac{\partial T}{\partial x}\right)^2\frac{\partial^2 k(T)}{\partial T\partial k_m}=C(T)\frac{\partial X_m}{\partial t} \quad 0<x<1 \quad t>0 \tag{7}$$

$$X_m(x,0)=0 \quad 0\leq x\leq 1 \quad t=0 \tag{8}$$

$$\frac{\partial k(T)}{\partial k_m}\frac{\partial T}{\partial x}+k(T)\frac{\partial X_m}{\partial x}=0 \quad x=0 \quad t>0 \tag{9}$$

$$\frac{\partial k(T)}{\partial k_m}\frac{\partial T}{\partial x}+k(T)\frac{\partial X_m}{\partial x}=0 \quad x=1 \quad t>0 \tag{10}$$

where

$$X_m=\frac{\partial T}{\partial k_m}$$

and

$$k(T)\frac{\partial^2 X_n}{\partial x^2}+\frac{\partial k(T)}{\partial T}\frac{\partial T}{\partial x}\frac{\partial X_n}{\partial x}-\frac{\partial C(T)}{\partial C_n}\frac{\partial T}{\partial t}=C(T)\frac{\partial X_n}{\partial t} \quad 0<x<1 \quad t>0 \tag{11}$$

$$X_n(x,0)=0 \quad 0\leq x\leq 1 \quad t=0 \tag{12}$$

$$k(T)\frac{\partial X_n}{\partial x}=0 \quad x=0 \quad t>0 \tag{13}$$

$$k(T)\frac{\partial X_n}{\partial x}=0 \quad x=1 \quad t>0 \tag{14}$$

where

$$X_n=\frac{\partial T}{\partial C_n}.$$

Eqs. (7)–(14) describe the mathematical equations for sensitivity coefficient X_m and X_n that can be explicitly found if $k(T)$, $C(T)$ and T are known. The equations are the linear equations and the dependent variable X_m and X_n are with respect to the independent variables x and t. Therefore, the sensitive data can be determined directly through a finite difference method.

2.3. A modified Newton–Raphson method

The Newton–Raphson method [16] has been widely adopted to solve a set of nonlinear equations. This method is applicable to solve the nonlinear problem when the number of the equations and the number of the unknown variables are the same. In the inverse problem, the number of equations is usually larger than the number of variables; therefore a modified version of the Newton–Raphson method is necessary to deal with the inverse problem.

In the present research, the proposed method formulates the problem from the comparison between the calculated temperature and the measured one directly. Therefore, the calculated temperature $\Phi_c(\bar{\imath}, j)$ and the measured temperature $\Phi_{meas}(\bar{\imath}, j)$ at the $\bar{\imath}$-grid of the spatial coordinate and at j-grid of the temporal coordinate are needed to be evaluated firstly. Then, the estimation of the unknown thermal conductivity and heat capacity can be recast as the solution of a set of nonlinear equations:

$$\Phi(\bar{\imath}, j)=\Phi_c(\bar{\imath}, j)-\Phi_{meas}(\bar{\imath}, j)=0. \tag{15}$$

where $j = 1, 2, 3, \ldots, N_t$ and N_t is the number of grid spacing along with the temporal coordinate.

This set of equations has $\bar{m}+\bar{n}$ variables. As well, the number of equations is the number of the temporal measurements when one sensor is used. If the number of independent equations is more than the number of the variables, the set of equation can be solved through the modified method. This detail procedure can be shown as follows:

Substitute the temporal index j from 1 to N_t into Eq. (15),

$$\mathbf{\Phi}=[\Phi(\bar{\imath},1),\ \Phi(\bar{\imath},2)\Phi(\bar{\imath},3),\ \ldots,\ \Phi(\bar{\imath},N_t)]^T=\{\hat{\Phi}_u\} \tag{16}$$

where $\bar{\Phi}_u$ is the component of vector $\mathbf{\Phi}$.

The undetermined coefficients are set as follows:

$$\mathbf{x}=[k_1,\ k_2,\ \ldots,\ k_{\bar{m}},\ C_1,\ C_2,\ \ldots,\ C_{\bar{n}}]^T=[x_1,\ x_2,\ x_3,\ x_4,\ \ldots,\ x_{\bar{m}+\bar{n}}]^T=\{x_v\} \tag{17}$$

where x_v is the component of vector $\mathbf{x}$.

The derivative of $\hat{\Phi}_u$ with respect to x_v is solved through Eqs. (7)–(14) and it can be expressed as follows:

$$\Psi_{u,v}=\frac{\partial\hat{\Phi}_u}{\partial x_v} \tag{18}$$

The sensitivity matrix $\mathbf{\Psi}$ can be defined as follows:

$$\mathbf{\Psi}=\{\Psi_{u,v}\} \tag{19}$$

where $u = 1, 2, 3, \ldots, N_t$ and $v = 1, 2, 3, \ldots, \bar{m}+\bar{n}$

and $\Psi_{u,v}$ is the element of $\mathbf{\Psi}$ at uth row and vth column.

The starting vector $\mathbf{x}_0$ can be shown in the following vector:

$$\mathbf{x}_0 = [x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, \ldots, x_{\bar{m}+\bar{n},0}]^{\mathrm{T}} \tag{20}$$

With the above derivations from Eqs (16) to (20), we have the following equation:

$$\mathbf{x}_{\lambda+1} = \mathbf{x}_\lambda + \Delta_\lambda \tag{21}$$

Δ_λ is a linear least-squares solution for a set of over-determined linear equations and it can be derived as follows:

$$\mathbf{\Psi}(\mathbf{x}_\lambda)\Delta_\lambda = -\mathbf{\Phi}(\mathbf{x}_\lambda) \tag{22}$$

$$\Delta_\lambda = -[\mathbf{\Psi}^{\mathrm{T}}(\mathbf{x}_\lambda)\mathbf{\Psi}(\mathbf{x}_\lambda)]^{-1}\mathbf{\Psi}^{\mathrm{T}}(\mathbf{x}_\lambda)\mathbf{\Phi}(\mathbf{x}_\lambda) \tag{23}$$

The above derivation is applied only at one-point measurement. However, this method can be implemented in the multi-sensors' measurement. Under this condition, the number of the elements in Eq. (16) is increased based on the formulation of Eq. (15). For example, if p sensors are used to measure the N_{t} number of the temporal grids, the number of the elements will be $p \times N_{\mathrm{t}}$ in Eq. (16). Moreover, the fundamental theorem concerning convergence of Eq. (23) is also shown in appendix.

2.4. The stopping criterion

The iterative process (Eqs. (21)–(23)) is used to determine the unknown vector $\mathbf{x}$ defined by Eq. (17). The step size Δ_λ goes from $\mathbf{x}_\lambda$ to $\mathbf{x}_{\lambda+1}$ and it is determined from Eq. (23). Once Δ_λ is calculated, the iterative process to determine $\mathbf{x}_{\lambda+1}$ is not stopped until the following criterion is satisfied

$$\mathbf{J}(\mathbf{x}_{\lambda+1}) < \epsilon \tag{24}$$

where

$$\mathbf{J}(\mathbf{x}_{\lambda+1}) = \frac{\Delta t}{N^{\mathrm{meas}} \times N_{\mathrm{t}}} \sum_{i=1}^{N^{\mathrm{meas}}} \sum_{j=1}^{N_{\mathrm{t}}} [\Phi_{\mathrm{c}}(\bar{i}, j) - \Phi_{\mathrm{meas}}(\bar{i}, j)]^2 \tag{25}$$

and ϵ is the value of stopping criterion and N^{meas} is the number of spatial measurements.

When measurement errors are not included, ϵ is a small specified number. However, the error-free measurement is difficult to achieve and then the measurement error is needed to be included in the numerical simulation. In other words, the value of $\mathbf{J}(\mathbf{x}_{\lambda+1})$ is not expected to vanish at the final step of iterations. Therefore, a discrepancy principle [1,6] is used to evaluate the value of the stopping criterion.

The temperature residual is approximated by,

$$\Phi_{\mathrm{meas}}(\bar{i}, j) - \Phi_{\mathrm{c}}(\bar{i}, j) \approx \sigma \tag{26}$$

where σ is the standard deviation of the measurements and it is assumed to be a constant in a certain experimental environment.

Substituting Eq. (26) into (25), the value of the stopping criterion ϵ has the following expression

$$\epsilon = \Delta t \sigma^2 \tag{27}$$

Then, the iterative process is terminated when Eq. (24) is satisfied by the value of ϵ.

3. Computational algorithm

The iterative procedure for the proposed method can be summarized as follows:

Given overall convergence tolerance $\epsilon > 0$ and the initial guess $\mathbf{x}_0$. The value of $\mathbf{x}_\lambda$ is known at the λth iteration.

Step 1. Solve the direct problem (Eqs. (3)–(6)), and compute the calculated temperature $\Phi_{\mathrm{c}}(\bar{i}, j)$

Step 2. Integrate the calculated temperature $\Phi_{\mathrm{c}}(\bar{i}, j)$ with the measured temperature $\Phi_{\mathrm{meas}}(\bar{i}, j)$ to construct $\mathbf{\Phi}$.

Step 3. Calculate the sensitivity matrix $\mathbf{\Psi}$ through Eqs. (7)–(14).

Step 4. Knowing $\mathbf{\Psi}$ and $\mathbf{\Phi}$, compute the step size Δ_λ from Eq. (23).

Step 5. Knowing Δ_λ and $\mathbf{x}_\lambda$, compute $\mathbf{x}_{\lambda+1}$ from Eq. (21).

Step 6. Terminate the process if the stopping criterion (Eq. (24)) is satisfied. Otherwise return to step 1.

4. Results and discussions

In this section, a problem with the specific thermophysical functions is used as an example to demonstrate the usage of the proposed method. In the example problem, the stability and the accuracy of the inverse estimation are testified. Furthermore, the results are also compared to the results of Huang and Yan's approach [14]. The exact temperature and the thermophysical properties used in the example are selected so that these functions can satisfy Eqs. (3)–(6). The accuracy of the proposed method is assessed by comparing the estimated results with the preselected thermophysical properties. Meanwhile, the measured temperature is generated from the preselected exact temperature in each problem and it is presumed to have measurement errors. In other words, the random errors of measurement are added to the exact temperature. It can be shown in the following equation:

$$T_{i,j}^{\text{meas}} = T_{i,j}^{\text{exact}} + \lambda\sigma \tag{28}$$

where $T_{i,j}^{\text{exact}}$ in Eq. (28) is the exact temperature. $T_{i,j}^{\text{meas}}$ is the measured temperature. σ is the standard deviation of measurement errors. λ is a random number.

The time domain is from 0 to 1.2 with 0.02 increment for the example problem. As well, the increment of spatial coordinate is 0.1. Four kinds of random noise level $\sigma = 0.001$, 0.005, 0.01 and 0.025 are adopted. The values of λ is calculated by the IMSL subroutine DRNNOR [17] and chosen over the range $-2.576 < \lambda < 2.576$, which represent the 99% confidence bound for the temperature measurement. Two thermocouples are allocated at the front and back surfaces of the medium; therefore, a total of 120 measurements are used to estimate the unknown values. The boundary heat fluxes are taken as $q_1 = 17$ and $q_2 = 6$ for the example problem. The inverse problem of estimating the thermal conductivity and the heat capacity have unique solution because the temperature is measured at two boundaries with prescribed heat flux. In the example problem, the initial guesses of the unknown variables are taken as one. Detailed descriptions for the example are shown as follows.

Example problem: The thermal conductivity is a sinusoidal-exponential form and the heat capacity is a polynomial function. The estimated thermophysical properties are assumed as follows:

$$\hat{k}(T) = 1 + 4.5 \times \exp\left(\frac{T}{80}\right) + 2.5 \times \sin\left(\frac{T}{3}\right) \tag{29}$$

$$\hat{C}(T) = 1.2 + 0.02T + 0.00001T^2. \tag{30}$$

For illustration, the estimated thermal conductivity and heat capacity are expressed as a general formulation. Both properties are approximated by a six-ordered power series ($\bar{m} = 7$ and $\bar{n} = 7$). A parameter κ is used to regularize the independent variable to avoid the unpredictable values of thermophysical properties.

$$k(\tau) = k_1 + k_2\tau + k_3\tau^2 + k_4\tau^3 + k_5\tau^4 + k_6\tau^5 + k_7\tau^6 \tag{31}$$

$$C(\tau) = C_1 + C_2\tau + C_3\tau^2 + C_4\tau^3 + C_5\tau^4 + C_6\tau^5 + C_7\tau^6 \tag{32}$$

where $\tau = T/\kappa$. The value of κ is set to twenty-five in the problem.

The temperature distribution at $x = 0.5$ is chosen to serve as the domain of the inverse solutions. Both properties have excellent approximations when measurement errors are free (see circular mark in Figs. 1 and 2). However, it is unrealistic because the error-free measurement is hard to achieve. Therefore, the measurement error is included in the numerical simulation. The simulated temperatures are obtained according to Eq. (28). Consequently, the inverse solutions are also shown in Figs. 1–4.

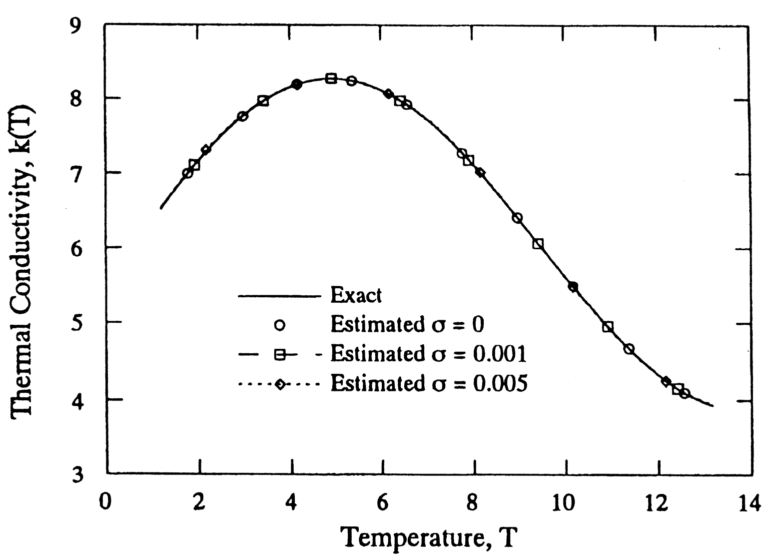

Fig. 1. The estimated thermal conductivity $k(T)$ when $\sigma = 0$, 0.001, and 0.005.

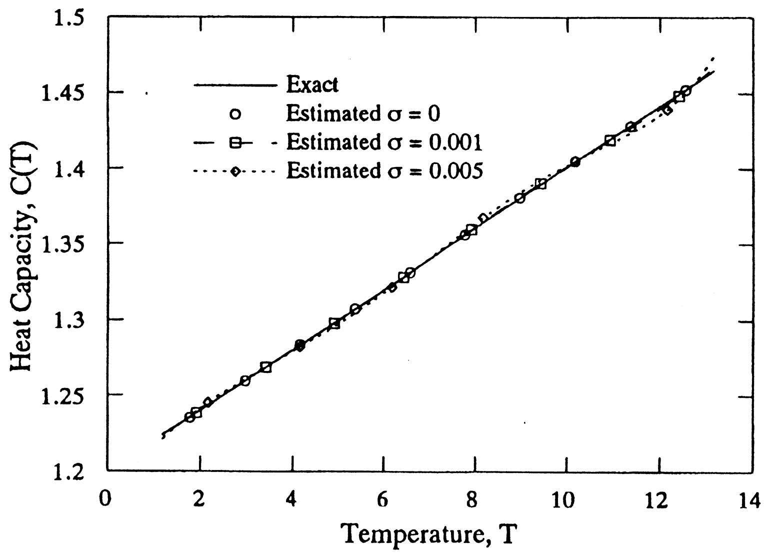

Fig. 2. The estimated heat capacity $C(T)$ when $\sigma = 0$, 0.001, and 0.005.

To further investigate the deviation of the estimated results from the exact solution, the relative average errors for the estimated properties are defined as follows:

$$\frac{1}{n_{\mathrm{T}}}\sum_{i=1}^{n_{\mathrm{T}}}\left|\frac{k(T_i)-\hat{k}(T_i)}{\hat{k}(T_i)}\right| \times 100\% \tag{33}$$

$$\frac{1}{n_{\mathrm{T}}}\sum_{i=1}^{n_{\mathrm{T}}}\left|\frac{C(T_i)-\hat{C}(T_i)}{\hat{C}(T_i)}\right| \times 100\% \tag{34}$$

where $k(T)$ and $\hat{k}(T)$ are denoted as the estimated and exact values of thermal conductivity. $C(T)$ and $\hat{C}(T)$ are denoted as the estimated and exact values of the heat capacity. n_{T} is the number of the interpolation points to calculate the relative average error.

To illustrate the characteristics of the present approach, the average relative errors and the number of iterations are shown in Table 1. When the value of σ is 0.001, the average relative errors of thermal conductivity and heat capacity are 0.0262% and 0.035% in the present approach and 1.502% and 0.706% in

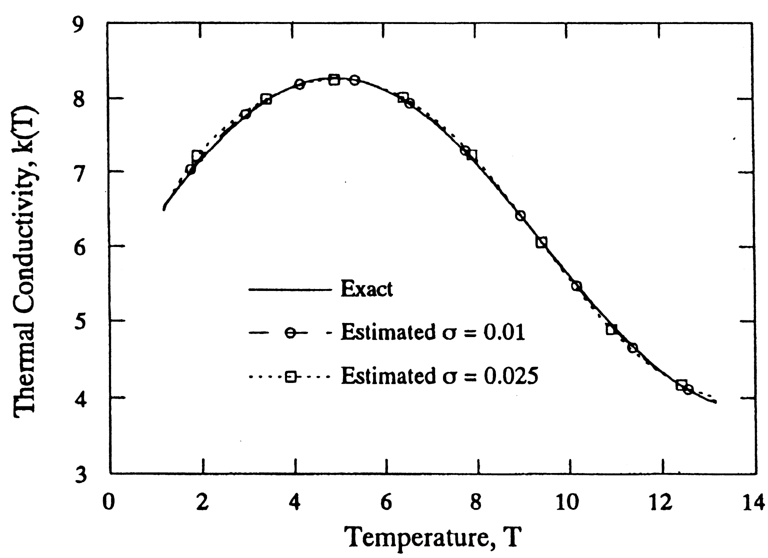

Fig. 3. The estimated thermal conductivity $k(T)$ when $\sigma = 0.01$ and 0.025.

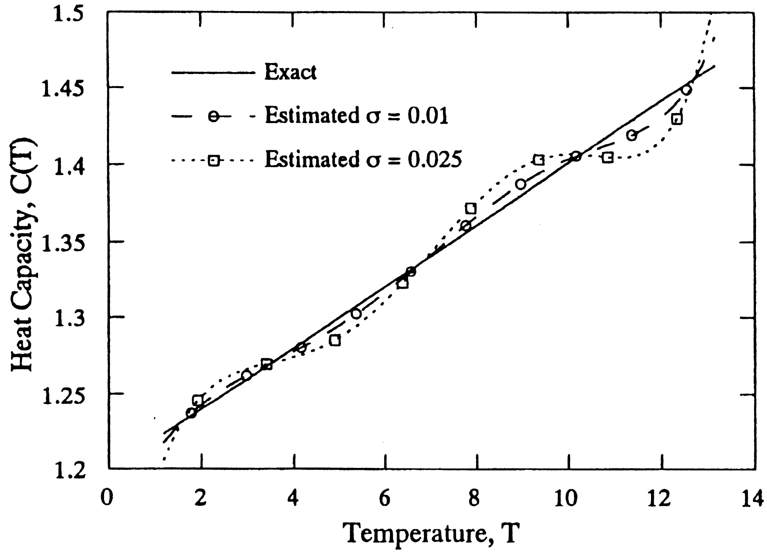

Fig. 4. The estimated heat capacity $C(T)$ when $\sigma = 0.01$ and 0.025.

Huang and Yan's results. Furthermore, when the value of σ is 0.005, the average relative errors of the estimation are 0.137% and 0.168% in the present approach and 4.101% and 1.026% in Huang and Yan's approach. It is clear that the present technique provides a more accurate and robust estimation. To further test the practicality of the proposed method, the value of the measurement error is raised to a realistic level. The results are shown in Figs. 3 and 4 for the thermal conductivity and the heat capacity when σ is 0.01 and 0.025. When $\sigma = 0.025$, the temperature error is within −0.0644–0.0644 for a 99% confidence bound, which implies that a maximum temperature error is 0.1288. The value of the measurement temperature is among 1.18–13.15 from the numerical simulation; thus, the average relative measurement error is about 1.8%. By using 1.8% measurement error, the average relative error is 0.705% in the thermal conductivity estimation and 0.831% in the heat capacity estimation. Furthermore, the means and variances of estimated error (i.e. estimated results minus exact function) are shown in Table 2. We compare the results of $k(T)$ and $C(T)$ when $\sigma = 0$, 0.001, 0.005, 0.01, and 0.025. The means and variances are small enough when measurement errors are appeared. It indicates that the proposed technique is suitable to estimate the thermal conductivity and the heat capacity simultaneously when a realistic error level is adopted. As well, the numerical values of the estimated results show that the measurement errors do not amplify the estimated errors. In other words, the proposed method provides a practical and confident prediction in estimating the thermal conductivity and the heat capacity simultaneously. Furthermore, the number of iterations to approach a satisfied result is below seven times, which shows that the speed of convergence of the propose method is fast. From the results of the example problem, it can be concluded that the proposed method is accurate and stable to simultaneously estimate the

Table 1
Convergent parameters for the example problem

Measurement error σ	Stopping criterion	No. of iterations	Average relative error $k(T)$ (%)		Average relative error $C(T)$ (%)	
			Present method	[14]	Present method	[14]
0	2.30×10^{-10}	7	0.012	0.510	0.012	0.690
0.001	2.00×10^{-8}	7	0.026	1.502	0.035	0.706
0.005	5.00×10^{-7}	6	0.137	4.101	0.168	1.026
0.01	2.00×10^{-6}	6	0.280		0.666	
0.025	1.25×10^{-5}	6	0.705		0.831	

Table 2
The means and variances of the error function

	$k(T)$, mean	$k(T)$, variance	$C(T)$, mean	$C(T)$, variance
0	−0.00006183	0.00000081	−0.00003869	0.00000004
0.001	0.00055819	0.00000321	−0.00007366	0.00000042
0.005	0.00278969	0.00009577	−0.00020802	0.00000787
0.01	0.00587211	0.00040395	−0.00036859	0.00003025
0.025	0.01505510	0.00260269	−0.00080776	0.00018068

thermal conductivity and the heat capacity in the inverse heat conduction problem.

5. Conclusion

An approach has been introduced to determine the thermal conductivity and the heat capacity simultaneously in the inverse heat conduction problem. The proposed method is constructed from a set of nonlinear equations from the measured temperatures and the calculated temperatures. An example has been illustrated based on the proposed method. Only two thermocouples are needed to measure the temperature at both sides of boundaries. Meanwhile, the speed of convergence of the proposed method is considerably fast. From the results, the average relative errors of thermal conductivity and heat capacity are considerably small for the example problem whether measurement error is included or excluded. The close agreement between the exact results and estimated results confirms that the proposed method is accurate and stable for the determination of the thermal conductivity and the heat capacity. The proposed method is applicable to the other kinds of inverse problems such as boundary estimation, initial estimation and source strength estimation in the one- or multi-dimensional inverse conduction problems.

Appendix

The following process shows the fundamental theorem concerning convergence, which is similar to the process in Ref. [16]. First, we have the result

$$\mathbf{\Phi}(\boldsymbol{\alpha}) = \mathbf{0}$$

when $\det(\mathbf{\Psi}^{\mathrm{T}}(\mathbf{x}_\lambda)\mathbf{\Psi}(\mathbf{x}_\lambda)) \neq 0$ and if the components of $\mathbf{\Psi}(\mathbf{x})$ are continuous in a neighborhood of a point α then

$$\lim_{\lambda\to\infty} \mathbf{x}_\lambda = \boldsymbol{\alpha} \quad \text{if} \quad \mathbf{x}_0 \quad \text{is near} \quad \alpha.$$

Therefore, Eq. (23) can be formed as follows:

$$\Delta_\lambda = (\mathbf{\Psi}^{\mathrm{T}}(\mathbf{x}_\lambda)\mathbf{\Psi}(\mathbf{x}_\lambda))^{-1}\mathbf{\Psi}^{\mathrm{T}}(\mathbf{x}_\lambda)(\mathbf{\Phi}(\boldsymbol{\alpha}) - \mathbf{\Phi}(\mathbf{x}_\lambda)).$$

By the mean-value theorem, we have

$$\hat{\Phi}_u(\mathbf{x}_\lambda) - \hat{\Phi}_u(\boldsymbol{\alpha}) = \sum_{v=1}^{\bar{m}+\bar{n}} \Psi_{u,\,v}(\boldsymbol{\alpha} + \xi_{u\lambda}(\mathbf{x}_\lambda - \boldsymbol{\alpha}))(x_{v,\,\lambda} - \alpha_v)$$

where $0 < \xi_{u\lambda} < 1$ and $\hat{\Phi}_u(\boldsymbol{\alpha}) = 0$.

Define a matrix $\mathbf{\Omega}$ with the components in the uth row:

$$[\Psi_{u,\,1}(\boldsymbol{\alpha} + \xi_{u\lambda}(\mathbf{x}_\lambda - \boldsymbol{\alpha})),\ \ldots,\ \Psi_{n,\,\bar{m}+\bar{n}}(\boldsymbol{\alpha} + \xi_{u\lambda}(\mathbf{x}_\lambda - \boldsymbol{\alpha}))].$$

Then we have the following equation

$$\begin{aligned}\mathbf{x}_{\lambda+1} - \boldsymbol{\alpha} &= \mathbf{x}_\lambda - \boldsymbol{\alpha} + \Delta_\lambda \\ &= (\mathbf{\Psi}^{\mathrm{T}}(\mathbf{x}_\lambda)\mathbf{\Psi}(\mathbf{x}_\lambda))^{-1}\mathbf{\Psi}^{\mathrm{T}}(\mathbf{x}_\lambda)(\mathbf{\Psi}(\mathbf{x}_\lambda) - \mathbf{\Omega})(\mathbf{x}_\lambda - \boldsymbol{\alpha}).\end{aligned}$$

Since the components in the matrix $\mathbf{\Psi}(\mathbf{x}_\lambda) - \mathbf{\Omega}$ are shown in the following form

$$\Psi_{u,\,v}(\mathbf{x}_\lambda) - \Psi_{u,\,v}(\boldsymbol{\alpha} + \xi_{u\lambda}(\mathbf{x}_\lambda - \alpha))$$

and the form can be kept uniformly small if the starting vector $\mathbf{x}_0$ lies in an initially chosen region $\mathbf{R}$ describable as $|x_v - \alpha_v| \leq h$.

Therefore, the property of the convergence in λ-iteration is $|x_{v,\lambda} - \alpha_v| \leq \mathrm{h}\mu^\lambda$, where $0 < \mu < 1$ and $1 \leq v \leq \bar{m}+\bar{n}$.

Thus the sequence $\{\mathbf{x}_\lambda\}$ converges to $\boldsymbol{\alpha}$.

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